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NOTES ON MAGNETO-HYDRODYNAMICS - NUMBER III

SPECIAL SOLUTIONS

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Preface

This note is an elaboration of the third note, "Transients," in the mimeographed series of 1954. We consider certain mathematically simple flows which are explicitly solvable and are chosen to exhibit certain basic properties of plasmas. In particular, we note that gyro and plasma oscillations are, in the simple case considered, manifestations of a single phenomenon which could be termed "gyro-plasma" oscillations. Elementary properties of the Debye shielding length, Larmor radius, and another quantity termed "magnetic shielding" length are discussed.

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Special Solutions

In this note we consider certain special flows which are simple enough to be obtained explicitly and which illuminate certain general fluid properties connected with the terms gyro (or Larmor) frequency, plasma frequency, and Debye length. They have the feature in common of dependence on a single independent variable, either time or space. In particular we shall conclude that gyro and plasma oscillations are different facets of a single phenomenon, rather than distinct phenomena.

1. Transients

First, let us consider a space-independent problem with time alone as the variable. From $\dot{B}$ + curl E = 0 we conclude that B is constant in time also. From div D = q we conclude that the medium is neutral. There remains

$$(1) \qquad \kappa \dot{E} = - \sum q_r u_r \quad , \quad \sum q_r = 0$$

as the only non-trivial one of Maxwell's equations. From the conservation of mass, we conclude that ρ_r and consequently q_r are constant in time. We take the momentum equations with frictional terms proportional to velocity,

$$(2) \qquad \rho_r \dot{u}_r = q_r(E + u_r \times B) - \sum_s \alpha_{rs} (u_r - u_s)$$

$$\alpha_{rs} = \alpha_{sr} > 0 \quad .$$

Since ρ_r, q_r, and B are constants, (1) and (2) are a set of linear ordinary differential equations for E and u_r as functions of time.

It is convenient to consider first the frictionless case, $\alpha_{rs} = 0$. Also, as a formal procedure, let us drop E in (2). We obtain

$$(3) \qquad \dot{u}_r = \omega_r \times u_r \ , \ \ \omega_r = -\gamma_r B$$

where $\gamma_r = q_r/\rho_r = \mathcal{E}_r/m_r$. The ω_r are the <u>gyro-frequencies</u>. Each fluid rotates at its gyro frequency, ω_r. More precisely, each fluid <u>translates</u> at a uniform velocity u_r which changes its direction in time. The velocity components parallel to B are constant. This fluid motion is, of course, closely related to the motion of an individual particle in a uniform magnetic field, namely as given by

$$m \frac{dv}{dt} = \mathcal{E} v \times B \quad .$$

A non-zero fluid velocity, u, results from a gas of rotating particles when the phases of the individual particles are not completely random.

Now, still with $\alpha_{rs} = 0$, we set B = 0 but keep E. Elimination of E between (1) and (2) yields

$$\rho_r \ddot{u}_r = -\frac{q_r}{\kappa} \sum_s q_s u_s \quad .$$

Multiplying by γ_r and summing yields

$$(4) \quad \begin{cases} \ddot{J} = -\Omega^2 J \\ \Omega^2 = \frac{1}{\kappa}\sum \gamma_r q_r = \frac{1}{\kappa}\sum \frac{\varepsilon r^2}{m_r} n_r \quad . \end{cases}$$

Ω is the plasma frequency. Writing $J = \mathcal{J} e^{i\Omega t}$, we have

$$(5) \quad \begin{cases} E = -J/i\kappa\Omega \\ u_r = \bar{u}_r + \gamma_r J/\kappa\Omega^2 \\ \sum q_r \bar{u}_r = 0 \end{cases}$$

The basic physical effect here is the inertia of the electric current, i.e., of the particles carrying it. The net fluid velocity is constant,

$$\sum \rho_r u_r = \sum \rho_r \bar{u}_r \quad .$$

As in the case of gyro oscillations, the motion of each fluid is a translation in a direction which varies in time. Considering the oscillating part only, all fluids are in phase. The polarization is arbitrary as compared to the previous case of gyro oscillations in which the polarization was circular.

If the oscillation is in a fixed direction, it can be considered to be the solution to the problem of an oscillating plane condenser discharge. In this case, the physical interpretation is somewhat more intuitive; there is a restoring force, E, set up by a separation of charges from the fluid into the condenser plates.

The conventional formula for plasma frequency is

(6) $$\Omega^2 = \frac{\epsilon^2 n}{\kappa m_-}$$

where m_- is the electron mass. This is an approximation to (4) obtained by considering electron mass small compared to any other mass. It is interesting to note that there is no separate plasma frequency for each fluid (as there is a gyro frequency); there is a single plasma frequency whose value is dominated by the particle of smallest mass.

Next we turn to the general case but still keep $a_{rs} = 0$. It is easy to see that the components of $\mathbf{u}_r$ and $\mathbf{E}$ parallel to B and perpendicular to B separate into two uncoupled sets of differential equations. The equations for the parallel components are identical to the plasma oscillations with B = 0 which were previously considered. Therefore, we need only consider the perpendicular components which we write

$$u_r = (u_r' \,;\, u_r'').$$

Eliminating $\mathbf{E}$ between (1) and (2),

(7) $$\ddot{u}_r = - \frac{\gamma_r}{\kappa} \sum_s q_s\, u_s + \gamma_r\, \dot{u}_r \times B \quad .$$

This system of equations can be written

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \begin{pmatrix} u' \\ u'' \end{pmatrix} = 0$$

where

$$\begin{cases} X \equiv D^2\ \delta_{rs} + \gamma_r q_s/\kappa \\ Y \equiv \gamma_r\ \mathbf{BD}\ \delta_{rs} \\ D = \frac{d}{dt}\ . \end{cases}$$

The transformation

$$\begin{pmatrix} Y^{-1} & X^{-1} \\ -X^{-1} & Y^{-1} \end{pmatrix} \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

$$A = Y^{-1}\,X + X^{-1}\,Y$$

reduces the problem to

$$A\,u' = 0$$

or

$$Y^{-1}XAu' = (Y^{-1}\,X + i\,I)\,(Y^{-1}\,X - iI)\,u' = 0, \quad I = \delta_{rs}\ .$$

From

$$Y^{-1}\,X = \frac{1}{\gamma_r BD}\,(D^2\ \delta_{rs} + \gamma_r\ q_s/\kappa)$$

we obtain for the frequencies, setting $D = i\omega$,

(8) $$|\,\omega(\omega \pm \omega_r)\ \delta_{rs} - \gamma_r\ q_s\,/\kappa\,| = 0, \quad \omega_r = -\gamma_r B$$

Only the - sign need be taken (changing the sign of ω_r merely changes the sign of ω), and a simple computation yields (dropping zero roots)

$$\omega = \frac{1}{\kappa}\sum_{r=1}^{n}\frac{\gamma_r q_r}{\omega - \omega_r}$$

or, in polynomial form,

$$(9) \qquad \kappa \prod_{r=1}^{n} (\omega - \omega_r) = \frac{1}{\omega} \sum_{r=1}^{n} \gamma_r q_r \prod_{s=1}^{n}{}_r (\omega - \omega_s)$$

where $\prod_{s=1}^{n}{}_r$ omits $s = r$. The right hand side is a polynomial since the sum is divisible by ω. For q_r approaching zero (low density) we obtain the gyro frequencies, $\omega = \omega_r$. For $B = 0$ ($\omega_r = 0$) we obtain the plasma frequency and the root $\omega = 0$ with multiplicity $2n - 2$. It is easy to expand in the neighborhood of $q_r = 0$ or $\omega_r = 0$. In the first case, we get

$$\omega^{(r)} = \omega_r + \frac{\gamma_r q_r}{\kappa \omega_r} + \ldots = \omega_r - \frac{q_r}{\kappa B} + \ldots$$

and, in particular

$$\sum \omega^{(r)2} = \sum \omega_r^2 + 2\Omega^2 + \ldots \quad .$$

In the second case, the double root Ω^2 splits,

$$\omega = \pm \Omega + \frac{1}{2} \sum \omega_r + \ldots$$

and the remaining roots are first order in ω_r and given by

$$\sum_{r=1}^{n} \frac{\gamma_r q_r}{\omega - \omega_r} = 0 \quad .$$

If one fluid consists of electrons and all others are heavy ions, $\gamma_1 \gg \gamma_2 \ldots \gamma_n$, $\omega_1 \gg \omega_2 \ldots \omega_n$, then there are two modes given by

$$\omega (\omega - \omega_1) = \Omega^2$$

while the remaining ones are of the order of magnitude of $\omega_2 \ldots \omega_n$.

We remark that the general solution consists of a superposition of plasma oscillations parallel to B and gyro-plasma oscillations perpendicular to B.

It is important to realize that, in the general case, the quantities ω_r and Ω^2 do not represent natural frequencies of the medium, but parameters from which the natural frequencies can be computed.

To treat the frictional terms in any sort of explicit form it is necessary to restrict the discussion to the case of two fluids. We have

$$(10)\quad \begin{cases} \kappa\dot{E} = -\,(q_1u_1 + q_2u_2) \\ \rho_1\dot{u}_1 = q_1\,(E + u_1 \times B) - a\,(u_1 - u_2) \\ \rho_2\dot{u}_2 = q_2\,(E + u_2 \times B) - a\,(u_2 - u_1) \\ q_1 + q_2 = 0 \end{cases}$$

Using

$$\rho_1\, u_1 + \rho_2\, u_2 = \rho\, u$$

$$q_1\, u_1 + q_2\, u_2 = J \quad ,$$

we introduce u and J instead of u_1 and u_2,

$$(11)\quad \begin{cases} \kappa\dot{E} = -\,J \\ \rho\dot{u} = J \times B \\ \dot{J} = \kappa\Omega^2\,(E + u \times B) + \dfrac{\gamma_1\rho_2 + \gamma_2\rho_1}{\rho}\, J\mathrm{x}B - \dfrac{a\rho}{\rho_1\rho_2}\,\dot{J} \end{cases} .$$

E and u can be eliminated,

$$(12)\quad \ddot{J} = -\,\Omega^2 J + \frac{\kappa\Omega^2}{\rho}\,(J\mathrm{x}B)\mathrm{x}B + \frac{\gamma_1\rho_2+\gamma_2\rho_1}{\rho}\,\dot{J}\mathrm{x}B - \frac{a\rho}{\rho_1\rho_2}\,\dot{J} \quad .$$

Parallel to B we have

(13) $$\ddot{J} + \omega_a \dot{J} + \Omega^2 J = 0, \quad \omega_d = \frac{a\rho}{\rho_1 \rho_2}$$

and perpendicular to B, introducing $\mathcal{J} = J \times B$,

(14) $$\ddot{\mathcal{J}} + \omega_a \dot{\mathcal{J}} + (\Omega^2 - \omega_1 \omega_2)\mathcal{J} + \frac{\gamma_1 \rho_2 + \gamma_2 \rho_1}{\rho} \dot{\mathcal{J}} \times B = 0$$

As before, solutions of (13) can be obtained as special cases of (14), taking B = 0 ($\omega_1 = \omega_2 = 0$).

The frequencies of equation (14) are given by

(15) $$\omega^2 + \omega[i\omega_a \pm (\omega_1 + \omega_2)] + \omega_1 \omega_2 - \Omega^2 = 0 \quad ,$$

from which their behavior can easily be seen. In the direction parallel to B we have

(16) $$\omega^2 + i\omega_a \omega - \Omega^2 = 0 \quad .$$

A non-oscillatory solution of (11) has J = 0 and $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$. The general solution consists of an oscillatory decay towards such a steady solution with damped plasma oscillations parallel to B and damped gyro-plasma oscillations perpendicular to B.

2. The Debye Length

We now consider the static fluid distribution which results from inserting a fixed point charge, say positive. The tendency will be to repel a positive fluid and attract a negative fluid. For simplicity, we linearize and assume that deviations from uniformity are small. The equilibrium equations are

(17) $$\begin{cases} \nabla p_r = q_r \mathbf{E} \\ \operatorname{div} E = q/\kappa \end{cases}$$

and, for a perfect gas,

(18) $$p_r = \nu_r kT_r = q_r kT/\varepsilon_r$$

so that, assuming T_r is constant,

(19) $$\nabla q_r = \frac{\varepsilon_r}{kT_r} \; q_r \, E \quad .$$

Summing, taking the divergence, and linearizing, we find

(20) $$\begin{cases} \Delta q = q/d^2 \\ \dfrac{1}{d^2} = \sum \dfrac{\varepsilon_r q_r}{\kappa k T_r} = \sum \dfrac{\varepsilon_r^{\,2} \; \nu_r}{\kappa k T_r} \end{cases} \quad ;$$

d is the <u>Debye length</u>. The solution for q is

(21) $$q = \alpha \, \frac{e^{-r/d}}{r/d} \quad .$$

The intuitive interpretation of d is a shielding distance; beyond a distance d from a given charge, the charge is not visible (E drops to zero exponentially). The fact that q becomes infinite as r ---> 0 violates the assumed linearization, but essentially the same results can be seen from the full nonlinear equations.

If the ε_r are equal and also the T_r ,

(22) $$\begin{cases} d^2 = \dfrac{\kappa kT}{\nu \varepsilon^2} \sim \dfrac{RT}{\Omega^2} \\ R = k/m \end{cases}$$

where m is the electron mass and the approximation (6) has been used. The connection between plasma frequency,

Debye length, and electron speed ($RT \sim \overline{v^2}$) would seem to be intuitively connected with the charge separation interpretation of plasma oscillations.

Another property of d comes from

$$(23) \qquad \begin{cases} kT = \dfrac{\nu\varepsilon^2 d^2}{\kappa} = \dfrac{\varepsilon(N\varepsilon)}{\kappa d} \\ N = \nu d^3 \end{cases}$$

The right hand side can be interpreted as the electrostatic energy between a charge ε and a charge $N\varepsilon$ (the total charge in a Debye cube) separated by a distance d. However, the potential energy assigned to a particle because of interactions with its neighbors, i.e., as given by the solution of (20), has the order of magnitude $\varepsilon^2/\kappa d$ since particles farther than d are ineffective, and the total separated charge is ε. We conclude that, to have a perfect gas ($\varepsilon^2/\kappa d \ll kT$), we must have $N \gg 1$; i.e., each particle must be in the potential field of many others! At normal temperature and pressure (which is artificial since there is no ionization) $N \sim 2 \times 10^{-4}$, while at a pressure of 1 mm and a temperature of 10,000°C, $N \sim 10$, which is a fairly good perfect gas.

A number of general estimates of orders of magnitude can be made using the lengths L_D and L_H defined in the last chapter. We compute

$$(24) \qquad \frac{q}{\nu\varepsilon} = d^2 \, \frac{q}{D} \, \frac{\varepsilon E}{kT} \sim \frac{d^2}{LL_D} \, \frac{\varepsilon EL}{kT}$$

Frequently εEL has the same order of magnitude as kT if L is the apparatus size. Furthermore, d is usually extremely small (2×10^{-8} cm. at normal temperature and pressure, 3×10^{-6} cm. at 10,000°C and 1 mm. pressure). We conclude that $q/\nu\varepsilon$ is small, i.e., the medium is neutral in the sense that the difference between the number of positive and negative particles is small compared to either. Conversely, if this is not so, the resulting E will be enormous compared to the particle energy. Alternatively, a small domain of dimension d over which E changes appreciably (e.g., in a boundary layer) may not be neutral.

An analogous computation can be made for the magnetic field,

$$(25)\quad \begin{cases} \dfrac{J}{\nu\varepsilon v} \sim \dfrac{H}{\nu\varepsilon v L_H} = \dfrac{HB}{m\nu v^2}\,\dfrac{\lambda}{L_H} = \dfrac{\delta}{L_H} \\ \lambda = \dfrac{mv}{\varepsilon B} = \dfrac{v}{\omega} \\ \delta = \dfrac{HB}{m\nu v^2}\,\lambda \quad ; \end{cases}$$

λ is the Larmor radius, namely, the radius of a particle orbit in a uniform magnetic field. First of all, since $J/\nu\varepsilon v$ cannot be larger than one in order of magnitude if v is a representative electron velocity, we find that L_H cannot be smaller than δ in order of magnitude. If $HB/m\nu v^2 \sim HB/p$ is of order one, the magnetic field cannot alter appreciably over a distance shorter than

the Larmor radius. Conversely, if L_H is large compared to δ, the relative velocity between ions and electrons is small compared to thermal velocities. Comparing with the previous example, we might call δ a magnetic shielding distance. The relation

$$\delta \sim \sqrt{\frac{HB}{p}}\,\frac{c}{v}\,d \tag{26}$$

is illuminating. In the case of electrostatic forces, d separates nearby, i.e., collisional type forces from long range forces. In the case of magnetic forces, δ accomplishes the same purpose. Taking HB/p to be of order one, $\delta \sim \frac{c}{v}\,d$, which, combined with the rough estimate of interparticle magnetic forces as being on the order of v/c smaller than electrostatic forces, implies that the "collisional" electrostatic and magnetic forces have equal orders of magnitude.